Some discussions on Hausdorff metric

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Abstract

In this paper, we discuss the properties of functions generated using Hausdorff metric.

Keywords: Fuzzy sets; Hausdorff metric; α-cut

1. Introduction

Let (X, d_X) be a metric space. For $u \in F(X)$, let $[u]_\alpha$ denote the α-cut of u, i.e.

$$[u]_\alpha = \begin{cases} \{x \in X : u(x) \geq \alpha\}, & \alpha \in (0, 1], \\ \operatorname{supp} u = \overline{\{u > 0\}}, & \alpha = 0, \end{cases}$$

where $\overline{S}$ denotes the closure of S in (X, d).

The set of upper semi-continuous fuzzy sets in (X, d_X) is denoted by $F_{USC}(X)$, i.e.

$$F_{USC}(X) = \{u \in F(X) : [u]_\alpha \text{ is closed in } (X, d_X) \text{ for } \alpha \in (0, 1]\}.$$

$F^1_{USC}(X)$ is the set of normal fuzzy sets in $F_{USC}(X)$, i.e.

$$F^1_{USC}(X) = \{u \in F(X) : [u]_\alpha \in C(X) \text{ for } \alpha \in (0, 1]\},$$

where $C(X)$ is the set of nonempty closed sets of (X, d_X).

$$F^1_{USCG}(X) = \{u \in F(X) : [u]_\alpha \in K(X) \text{ for } \alpha \in (0, 1]\},$$

where $K(X)$ is the set of nonempty compact sets of (X, d_X).

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Let (X, d_X) be a metric space. We use $\boldsymbol{H}$ to denote the ***Hausdorff metric*** on $C(X)$ induced by d_X, i.e.,

$$\boldsymbol{H(U,V)} = \max\{H^*(U,V),\ H^*(V,U)\} \tag{1}$$

for arbitrary $U, V \in C(X)$, where

$$H^*(U,V) = \sup_{u\in U} d_X\,(u, V) = \sup_{u\in U} \inf_{v\in V} d_X\,(u,v).$$

Remark 1.1. ρ is said to be a extended metric on Y if ρ is a function from $Y \times Y$ into $\mathbb{R} \cup \{+\infty\}$ satisfying positivity, symmetry and triangle inequality. The Hausdorff metric H on $C(X)$ induced by d_X on X is a extended metric, but probably not a metric, because $H(A,B)$ could be equal to $+\infty$ for some $A, B \in C(X)$.

The following inequality should be a known result.

$$H^*(U,W) \le H^*(U,V) + H^*(V,W) \tag{2}$$

for $U, V, W \in C(X)$.

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space. See [1] for the symbols in this paper.

In this paper, we uniformly use H to denote the Hausdorff metric on $C(X)$ induced by d_X, where (X, d_X) is a certain metric space. The meaning of H can be judged according to the context.

We have obtained the following statements on the measurability of the function $H([u]_\alpha, [v]_\alpha)$ (See [3], which was submitted on 2019.07.06).

- For $u \in F^1_{USC}(X)$ and $x_0 \in X$, $H([u]_\alpha, \{x_0\})$ is a measurable function of α on $[0,1]$.
- For $u, v \in F^1_{USC}(\mathbb{R}^m)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.
- For $u, v \in F^1_{USCG}(X)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.
- There exists a metric space X and $u, v \in F^1_{USC}(X)$ such that $H([u]_\alpha, [v]_\alpha)$ is a non-measurable function of α on $[0,1]$.

In [6], we submitted the proofs of the first three statements.

The proofs of the first three statements and the example given in chinaXiv:202108.00116v1 which shows the last statement were recorded in a handwritten material before 2019.07.06. In this version, a very small change is made to the example.

2. Properties of $H([u]_\alpha, [v]_\alpha)$

In this section, we give the proofs of the first three statements and the example to show the last statement.

Proposition 2.1. *For $u \in F^1_{USC}(X)$ and $x_0 \in X$, $H([u]_\alpha, \{x_0\})$ is a measurable function of α on $[0, 1]$.*

Proof. We can see that for $0 \le \alpha \le \beta \le 1$,

$$H([u]_\alpha, \{x_0\}) = \sup_{x\in[u]_\alpha} d(x, x_0) \ge \sup_{x\in[u]_\beta} d(x, x_0) = H([u]_\beta, \{x_0\}).$$

So the desired result follows from the fact that $H([u]_\alpha, \{x_0\})$ is a monotone function of α on $[0, 1]$.

□

For $u, v \in F^1_{USC}(X)$ and $r \in \mathbb{R}$, we use the symbol $\{H(u, v) > r\}$ to denote the set $\{\alpha \in [0, 1] : H([u]_\alpha, [v]_\alpha) > r\}$.

Proposition 2.2. *For $u, v \in F^1_{USC}(\mathbb{R}^m)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$.*

Proof. We only need to show that for each $r \in \mathbb{R}$, the set $\{H(u, v) > r\}$ is measurable set.

Step (i) For each $r \in \mathbb{R}$, if $\alpha > 0$ and $\alpha \in \{H(u, v) > r\}$, then there exists $\delta(\alpha) > 0$ such that $[\alpha - \delta(\alpha), \alpha] \subseteq \{H(u, v) > r\}$.

We proceed by contradiction. If for each $\delta > 0$, $[\alpha-\delta, \alpha] \nsubseteq \{H(u, v) > r\}$. Then there exists an increasing sequence $\{\gamma_n\}$ such that $\gamma_n \to \alpha$ and

$$H([u]_{\gamma_n}, [v]_{\gamma_n}) \le r.$$

Given $x \in [u]_\alpha$, then $d(x, [v]_{\gamma_n}) \le H([u]_{\gamma_n}, [v]_{\gamma_n}) \le r$. Therefore there exist $y_n \in [v]_{\gamma_n}$ such that $d(x, y_n) = d(x, [v]_{\gamma_n}) \le r$. Hence there is a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $\{y_{n_i}\}$ converges to $y \in \mathbb{R}^m$. Note that $d(x, y) \le r$ and $y \in \cap[v]_{\gamma_{n_i}} = [v]_\alpha$, so we have $d(x, [v]_\alpha) \le r$.

From the arbitrariness of x, $H^*([u]_\alpha, [v]_\alpha) \le r$. Similarly, we can deduce that $H^*([v]_\alpha, [u]_\alpha) \le r$. Thus $H([u]_\alpha, [v]_\alpha) \le r$, which is a contradiction.

Step (ii) For each $r \in \mathbb{R}$, if $\{H(u, v) > r\} \setminus \{0\} \neq \emptyset$, then $\{H(u, v) > r\} \setminus \{0\}$ is a union of disjoint positive length intervals.

Suppose that $\{H(u,v)>r\}\setminus\{0\}\neq\emptyset$. For $x\in\{H(u,v)>r\}\setminus\{0\}$, let $\overbrace{x}=\bigcup\{[a,b]: x\in[a,b]\subseteq\{H(u,v)>r\}\setminus\{0\}\}$, i.e. $\overbrace{x}$ is the largest interval in $\{H(u,v)>r\}\setminus\{0\}$ which contains x. Then by step (i), $\overbrace{x}$ is a positive length interval. Note that for $x,y\in\{H(u,v)>r\}\setminus\{0\}$, if $\overbrace{x}\cap\overbrace{y}\neq\emptyset$, then $\overbrace{x}=\overbrace{y}$. Thus $\{H(u,v)>r\}\setminus\{0\}$ is a union of disjoint positive length intervals.

Step (iii) For each $r\in\mathbb{R}$, $\{H(u,v)>r\}$ is a measurable set.

Clearly, if positive length intervals are disjoint, then these positive length intervals are at most countable. Thus, from step (ii), $\{H(u,v)>r\}\setminus\{0\}$ is a measurable set. So $\{H(u,v)>r\}$ is a measurable set. □

Remark 2.3. Let $u,v\in F^1_{USC}(X)$. For each $r\in\mathbb{R}$, if $0\in\{H(u,v)>r\}$, then there exists $\delta>0$ such that $[0,\delta]\subseteq\{H(u,v)>r\}$.

The above fact is equivalent to the following fact

Let $u,v\in F^1_{USC}(X)$. Then $H([u]_0,[v]_0)\leq\liminf_{\alpha\to0+}H([u]_\alpha,[v]_\alpha)$, here $\liminf_{\alpha\to0+}H([u]_\alpha,[v]_\alpha)=+\infty$ is possible.

Combined this fact with the proof of Proposition 2.2, we have the following conclusion

Let $u,v\in F^1_{USC}(\mathbb{R}^m)$ and let $r\in\mathbb{R}$. If $\{H(u,v)>r\}\neq\emptyset$, then $\{H(u,v)>r\}$ is a union of disjoint positive length intervals (Obviously, $\{H(u,v)>r\}$ could be an interval. It is easy to see that for fixed $r\geq0$, the possible forms of the maximal intervals in $\{H(u,v)>r\}$ are as follows: $[0,\alpha)$, $[0,\alpha]$, (β,α) and $(\beta,\alpha]$, where $\alpha\in(0,1]$ and $\beta\in[0,1)$).

Remark 2.4. Let $u,v\in F^1_{USC}(X)$ and let $\alpha>0$. The following two statements are equivalent.
(i) For each $r\in\mathbb{R}$, if $\alpha\in\{H(u,v)>r\}$, then there exists $\delta(\alpha)>0$ such that $[\alpha-\delta(\alpha),\alpha]\subseteq\{H(u,v)>r\}$.
(ii) $H([u]_\alpha,[v]_\alpha)\leq\liminf_{\gamma\to\alpha-}H([u]_\gamma,[v]_\gamma)$ ($\liminf_{\gamma\to\alpha-}H([u]_\gamma,[v]_\gamma)=+\infty$ is possible).

So the statement "$H([u]_\alpha,[v]_\alpha)\leq\liminf_{\gamma\to\alpha-}H([u]_\gamma,[v]_\gamma)$ for all $\alpha\in(0,1]$" is equivalent to the statement proved by step (i) of the proof of Proposition 2.2, which is listed below

- For each $r\in\mathbb{R}$, if $\alpha>0$ and $\alpha\in\{H(u,v)>r\}$, then there exists $\delta(\alpha)>0$ such that $[\alpha-\delta(\alpha),\alpha]\subseteq\{H(u,v)>r\}$.

Remark 2.5. From the proof of Proposition 2.2 and Remark 2.4, we know that for $u, v \in F^1_{USC}(X)$, if $H([u]_\alpha, [v]_\alpha) \leq \liminf_{\gamma\to\alpha-} H([u]_\gamma, [v]_\gamma)$ for all $\alpha \in (0, 1]$, then $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$.

The following Proposition 2.6 is Lemma 4.4 in [1].

Proposition 2.6. *Let $U_n \in K(X)$ for $n = 1, 2, \ldots$.*
(i) If $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots$, then $U = \bigcap_{n=1}^{+\infty} U_n \in K(X)$ and $H(U_n, U) \to 0$ as $n \to +\infty$.
(ii) If $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_n \subseteq \ldots$ and $V = \overline{\bigcup_{n=1}^{+\infty} U_n} \in K(X)$, then $H(U_n, V) \to 0$ as $n \to +\infty$.

Proof. (i) is easy to show. We only prove (ii). Suppose that $H(U_n, U) \not\to 0$. Then there is an $\varepsilon_0 > 0$ such that $H(U_n, U) > \varepsilon_0$. Hence there exists $x_n \in U$ such that

$$d(x_n, U_n) > \varepsilon_0. \tag{3}$$

Since U is compact, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that x_{n_i} converges to $x \in U$. Note that there exists $\{y_n\}$ such that $y_n \in U_n$ and $y_n \to x$. Thus $d(x, U_n) \to 0$, which contradicts (3). □

Proposition 2.7. *For $u, v \in F^1_{USCG}(X)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$.*

Proof. Note that $H([u]_\alpha, [v]_\alpha)$ is finite at $\alpha \in (0, 1]$ and for $\alpha, \beta \in (0, 1]$,

$$|H([u]_\alpha, [v]_\alpha) - H([u]_\beta, [v]_\beta)| \leq H([u]_\alpha, [u]_\beta) + H([v]_\alpha, [v]_\beta).$$

Then by Proposition 2.6 (i),

$$\lim_{\beta\to\alpha-} H([u]_\beta, [v]_\beta) = H([u]_\alpha, [v]_\alpha),$$

i.e. $H([u]_\alpha, [v]_\alpha)$ is left-continuous at $\alpha \in (0, 1]$.

Thus from Remark 2.5, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$. □

Remark 2.8. From Proposition 2.6, we know that for $u, v \in F^1_{USCG}(X)$, $H([u]_\alpha, [v]_\alpha)$ is left-continuous at $\alpha \in (0, 1]$, and that for $u, v \in F^1_{USCB}(X)$, $H([u]_\alpha, [v]_\alpha)$ is right-continuous at $\alpha = 0$.

To give the example which shows the last statement presented in Section 1, we need some conclusions at first.

The following representation theorem should be a known conclusion.

Theorem 2.9. *Let X be a set. Given $u \in F(X)$, then for all $\alpha \in (0,1]$, $[u]_\alpha = \cap_{\beta<\alpha}[u]_\beta$.*

Conversely, suppose that $\{u(\alpha) : \alpha \in (0,1]\}$ is a family of sets in X satisfying $u(\alpha) = \cap_{\beta<\alpha}u(\beta)$ for all $\alpha \in (0,1]$. Define $v \in F(X)$ by $v(x) := \sup\{\alpha : x \in u(\alpha)\}$ ($\sup \emptyset = 0$). Then $[v]_\alpha = u(\alpha)$ for all $\alpha \in (0,1]$.

ρ is said to be a *metric* on Y if ρ is a function from $Y \times Y$ into $\mathbb{R}$ satisfying positivity, symmetry and triangle inequality. At this time, (Y,ρ) is said to be a metric space.

ρ is said to be an *extended metric* on Y if ρ is a function from $Y \times Y$ into $\mathbb{R} \cup \{+\infty\}$ satisfying positivity, symmetry and triangle inequality. At this time, (Y,ρ) is said to be an extended metric space.

Let (Y,ρ) be an extended metric space. For $y \in Y$ and $\varepsilon > 0$, let $B(y,\varepsilon)$ denote the set $\{z \in Y : \rho(y,z) < \varepsilon\}$. $\{B(y,\varepsilon) : y \in Y, \varepsilon > 0\}$ is a basis for the topology induced by ρ on Y. The closure of a set A in (Y,ρ), denoted by $\overline{A}$, refers to the closure of A in Y according to the topology induced by ρ on Y. Then $x \in \overline{A}$ if and only if there is a sequence $\{x_n\}$ in Y such that $\rho(x_n,x) \to 0$. So $x \in \overline{A}$ if and only if $\rho(x,A) = 0$.

Here we mention that if (Y,ρ) is an extended metric space, then the Hausdorff distance H on $C(Y)$ induced by ρ using (1) is an extended metric on $C(Y)$, where $C(Y)$ denotes the set of nonempty closed sets in (Y,ρ). It can be seen that H satisfies positivity and symmetry. To show that H satisfies the triangle inequality, we only need to show that

$$H^*(U,W) \le H^*(U,V) + H^*(V,W) \tag{4}$$

for $U,V,W \in C(Y)$. To do this, let $x \in U$. Then

$$\begin{aligned}
\rho(x,W) &\le \inf_{y\in V}\inf_{z\in W}\{\rho(x,y)+\rho(y,z)\} \\
&\le \inf_{y\in V}\{\rho(x,y)+\rho(y,W)\} \\
&\le \inf_{y\in V}\rho(x,y) + H^*(V,W) \\
&= \rho(x,V) + H^*(V,W) \\
&\le H^*(U,V) + H^*(V,W).
\end{aligned}$$

From the arbitrariness of x in U, we obtain (4). So the Hausdorff distance H on $C(Y)$ is the Hausdorff extended metric.

For simplicity, we refer to both the Hausdorff extended metric and the Hausdorff metric as the Hausdorff metric in this paper.

For an extended metric space (Y, ρ), we define

$$F_{USC}(Y) = \{u \in F(Y) : [u]_\alpha \text{ is closed in } (Y, \rho) \text{ for } \alpha \in (0,1]\}.$$

Let Γ be a set, and for each $\gamma \in \Gamma$, let (X_γ, d_γ) be a metric space. Define an extended metric d on $\prod_{\gamma\in\Gamma} X_\gamma$ as

$$d(x, y) := \sup\{d_\gamma(x_\gamma, y_\gamma) : \gamma \in \Gamma\} \tag{5}$$

for $x = (x_\gamma)_{\gamma\in\Gamma}$ and $y = (y_\gamma)_{\gamma\in\Gamma}$.

We use the symbol $\prod_{\gamma\in\Gamma}(X_\gamma, d_\gamma)$ to denote the extended metric space $(\prod_{\gamma\in\Gamma} X_\gamma, d)$. If not mentioned specially, we suppose by default that the extended metric on $\prod_{\gamma\in\Gamma} X_\gamma$ is the d given by (5).

Let $u_\gamma \in F(X_\gamma)$, $\gamma \in \Gamma$. Define $u \in F(\prod_{\gamma\in\Gamma} X_\gamma)$ as

$$[u]_\alpha = \prod_{\gamma\in\Gamma} [u_\gamma]_\alpha \text{ for each } \alpha \in (0,1]. \tag{6}$$

We use $\boldsymbol{\prod_{\gamma\in\Gamma} u_\gamma}$ to denote the fuzzy set u given by (6).

From Theorem 2.9, u is well-defined because for each $\alpha \in (0,1]$,

$$[u]_\alpha = \prod_{\gamma\in\Gamma}[u_\gamma]_\alpha = \bigcap_{\beta<\alpha}\prod_{\gamma\in\Gamma}[u_\gamma]_\beta = \bigcap_{\beta<\alpha}[u]_\beta.$$

In this paper, if not mentioned specially, we use $\overline{S}$ to denote the closure of S in a certain extended metric space (X, d_X). For a set $S \subseteq X_\gamma$, $\gamma \in \Gamma$, we use $\overline{S}$ to denote the closure of S in (X_γ, d_γ). For a set $S \subseteq \prod_{\gamma\in\Gamma} X_\gamma$, we also use $\overline{S}$ to denote the closure of S in $(\prod_{\gamma\in\Gamma} X_\gamma, d)$. The readers can judge the meaning of $\overline{S}$ according to the context.

Lemma 2.10. *Let Γ be a set, and for each $\gamma \in \Gamma$, let (X_γ, d_γ) be a metric space. If $A_\gamma \subseteq X_\gamma$ for $\gamma \in \Gamma$, then $\overline{\prod_{\gamma\in\Gamma} A_\gamma} = \prod_{\gamma\in\Gamma} \overline{A_\gamma}$.*

Proof. Clearly $\overline{\prod_{\gamma\in\Gamma} A_\gamma} \subseteq \prod_{\gamma\in\Gamma} \overline{A_\gamma}$.

Conversely, if $x = (x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} \overline{A_\gamma}$, then for each $\varepsilon > 0$, there exists $y = (y_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} A_\gamma$ such that $d_\gamma(x_\gamma, y_\gamma) \le \varepsilon$ for all $\gamma \in \Gamma$. So $d(x, y) \le \varepsilon$. From the arbitrariness of $\varepsilon > 0$, we have $x \in \overline{\prod_{\gamma\in\Gamma} A_\gamma}$. Thus $\overline{\prod_{\gamma\in\Gamma} A_\gamma} \supseteq \prod_{\gamma\in\Gamma} \overline{A_\gamma}$.

In summary, $\overline{\prod_{\gamma\in\Gamma} A_\gamma} = \prod_{\gamma\in\Gamma} \overline{A_\gamma}$.

$\square$

Theorem 2.11. *Let Γ be a set, and for each $\gamma \in \Gamma$, let (X_γ, d_γ) be a metric space. If $u_\gamma \in F_{USC}(X_\gamma)$ for each $\gamma \in \Gamma$, then $u = \prod_{\gamma\in\Gamma} u_\gamma$ is a fuzzy set in $F_{USC}(\prod_{\gamma\in\Gamma} X_\gamma)$.*

Proof. By (6) and Lemma 2.10, for each $\alpha \in (0, 1]$,

$$\overline{[u]_\alpha} = \prod_{\gamma\in\Gamma} \overline{[u_\gamma]_\alpha} = \prod_{\gamma\in\Gamma} [u_\gamma]_\alpha = [u]_\alpha,$$

thus $u \in F_{USC}(\prod_{\gamma\in\Gamma} X_\gamma)$.

$\square$

In the following theorem, we use H to denote the Hausdorff metric on $C(X_\gamma)$ induced by d_γ. We also use H to denote the Hausdorff metric on $C(\prod_{\gamma\in\Gamma} X_\gamma)$ induced by d.

Theorem 2.12. *Let Γ be a set, and for each $\gamma \in \Gamma$, let (X_γ, d_γ) be a metric space. If A_γ and B_γ are elements in $C(X_\gamma)$ for $\gamma \in \Gamma$, then $H(\prod_{\gamma\in\Gamma} A_\gamma, \prod_{\gamma\in\Gamma} B_\gamma) = \sup_{\gamma\in\Gamma} H(A_\gamma, B_\gamma)$.*

Proof. From Lemma 2.10, $\prod_{\gamma\in\Gamma} A_\gamma$ and $\prod_{\gamma\in\Gamma} B_\gamma$ are elements in $C(\prod_{\gamma\in\Gamma} X_\gamma)$.

Note that $d(x, \prod_{\gamma\in\Gamma} B_\gamma) = \sup_{\gamma\in\Gamma} d_\gamma(x_\gamma, B_\gamma)$ for each $x = (x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X_\gamma$. Thus

$$\begin{aligned} H^*(\prod_{\gamma\in\Gamma} A_\gamma, \prod_{\gamma\in\Gamma} B_\gamma) &= \sup_{x\in\prod_{\gamma\in\Gamma} A_\gamma} d(x, \prod_{\gamma\in\Gamma} B_\gamma) \\ &= \sup_{x\in\prod_{\gamma\in\Gamma} A_\gamma} \sup_{\gamma\in\Gamma} d_\gamma(x_\gamma, B_\gamma) \\ &= \sup_{\gamma\in\Gamma} \sup_{x_\gamma\in A_\gamma} d_\gamma(x_\gamma, B_\gamma) \\ &= \sup_{\gamma\in\Gamma} H^*(A_\gamma, B_\gamma). \end{aligned}$$

So

$$H(\prod_{\gamma\in\Gamma} A_\gamma, \prod_{\gamma\in\Gamma} B_\gamma) = \sup_{\gamma\in\Gamma} H(A_\gamma, B_\gamma).$$

$\square$

Now, we give an example to show that there exists a metric space X and $u, v \in F^1_{USC}(X)$ such that $H([u]_\alpha, [v]_\alpha)$ is a non-measurable function of α on $[0, 1]$.

Example 2.13. We see $[0,100]\setminus\{10\}$ as a metric subspace of $\mathbb{R}$. Let $z\in(0,1]$. Define $u^z\in F^1_{USC}([0,100]\setminus\{10\})$ as

$$[u^z]_\alpha=\begin{cases}\{3\}, & \alpha\in[z,1],\\ \{3\}\cup(10,10+\varepsilon], & \alpha=z(1-\varepsilon),\ 0<\varepsilon\le 1.\end{cases}$$

Let $z\in(0,1]$. Define $v^z\in F^1_{USC}([0,100]\setminus\{10\})$ as

$$[v^z]_\alpha=\begin{cases}\{73\}, & \alpha\in(z,1],\\ [71,81], & \alpha\in[0,z].\end{cases}$$

Then for $z\in(0,1]$,

$$H([u^z]_\alpha,[v^z]_\alpha)=\begin{cases}70, & \alpha\in(z,1],\\ 78, & \alpha=z,\\ 71-\varepsilon, & \alpha=z(1-\varepsilon),\ 0<\varepsilon\le 1,\end{cases}\tag{7}$$

where H is the Hausdorff metric on $C([0,100]\setminus\{10\})$ induced by the metric on $[0,100]\setminus\{10\}$.

We see $[0,9]$ as a metric subspace of $\mathbb{R}$. Define $w\in F([0,9])$ as $w(t)=1$ for all $t\in[0,9]$.

Let A be a non-measurable set in $(0,1]$.

Let $u:=\prod_{z\in[0,1]}u_z$ and let $v:=\prod_{z\in[0,1]}v_z$, where

$$u_z=\begin{cases}u^z, & z\in A,\\ w, & z\in[0,1]\setminus A,\end{cases}\quad v_z=\begin{cases}v^z, & z\in A,\\ w, & z\in[0,1]\setminus A.\end{cases}$$

Then by Theorem 2.11, u and v are fuzzy sets in $F^1_{USC}(\prod_{z\in[0,1]}X_z)$, where

$$X_z=\begin{cases}[0,100]\setminus\{10\}, & z\in A,\\ [0,9], & z\in[0,1]\setminus A.\end{cases}$$

Here we mention that $(\prod_{z\in[0,1]}X_z,d)$ is a metric space with d given by (5).

By Theorem 2.12,

$$\begin{aligned}&H([u]_\alpha,[v]_\alpha)\\ &=\sup_{z\in A}H([u^z]_\alpha,[v^z]_\alpha)\vee\sup_{z\in[0,1]\setminus A}H([0,9],[0,9])\\ &=\sup_{z\in A}H([u^z]_\alpha,[v^z]_\alpha)\\ &\begin{cases}=78, & \alpha\in A,\\ \le 71, & \alpha\in[0,1]\setminus A.\end{cases}\end{aligned}$$

So $\{\alpha\in[0,1]:H([u]_\alpha,[v]_\alpha)>73\}=A$, and thus $H([u]_\alpha,[v]_\alpha)$ is a non-measurable function of α on $[0,1]$.

3. Some discussions

In [3] (Lemma 6.3) and [1] (Lemma 6.5), we pointed out that for $u \in F^1_{USCG}(X)$, the cut-function $[u](\alpha) = [u]_\alpha$ from $[0,1]$ to $(C(X), H)$ is left-continuous on $(0,1]$. Then it follows immediately that for $u, v \in F^1_{USCG}(X)$, $H([u]_\alpha, [v]_\alpha)$ is left-continuous at $\alpha \in (0,1]$ (see Proposition 2.7). From this fact, it's natural to realize that for $u, v \in F^1_{USCG}(X)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.

Let (X, d_X) be a metric space. We say that $S \subseteq F^1_{USC}(X)$ satisfies condition (X, d_X)-I if $[u]_\alpha \cap B(x,r)$ is compact in (X, d_X) for all $u \in S$, $\alpha \in (0,1]$, $x \in X$ and $r \in \mathbb{R}^+$, where $B(x,r) := \{y \in X : d_X(x,y) \le r\}$.

Clearly, $S = F^1_{USC}(\mathbb{R}^m)$ satisfies condition $\mathbb{R}^m$-I and $S = F^1_{USCG}(X)$ satisfies condition (X, d_X)-I.

If $S \subseteq F^1_{USC}(X)$ satisfies condition (X, d_X)-I, then proceed similarly as the step (i) of the proof of Proposition 2.2, we have that $H([u]_\alpha, [v]_\alpha) \le \liminf_{\gamma \to \alpha-} H([u]_\gamma, [v]_\gamma)$ for all $u, v \in S$ and $\alpha \in (0,1]$. Thus as mentioned in Remark 2.5, for all $u, v \in S$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.

There exists metric space (X, d_X) and $S \subseteq F^1_{USC}(X)$ which satisfies a condition weaker than condition (X, d_X)-I. By using this weaker condition, we can proceed similarly as the step (i) of the proof of Proposition 2.2 to show that $H([u]_\alpha, [v]_\alpha) \le \liminf_{\gamma \to \alpha-} H([u]_\gamma, [v]_\gamma)$ for all $u, v \in S$ and $\alpha \in (0,1]$.

4. Improvements

In this section, we give some improvements of Propositions 2.1, 2.2 and 2.7, which are the statements on measurability of $H([u]_\alpha, [v]_\alpha)$ presented in [1]. We first prove Theorem 4.1 which is an improvement of Propositions 2.1 and 2.7. Then we show Theorem 4.3 and use it to improve Theorem 4.1 and Proposition 2.2.

Let $v \in F^1_{USC}(X)$ and let $0 \le \alpha < \beta \le 1$. The "variation" $w_v(\alpha, \beta)$ is defined as $w_v(\alpha, \beta) := \sup\{H([v]_\xi, [v]_\eta) : \xi, \eta \in (\alpha, \beta]\}$.

Theorem 4.1. *Let $u \in F^1_{USC}(X)$ and let $v \in F^1_{USCG}(X)$. Then $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.*

Proof. The proof is divided into three steps.

Step (I) $H^*([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.

Let $\xi \in \mathbb{R}$ and let $n \in \mathbb{N}$. Define

$$S_\xi := \{\alpha \in [0,1] : H^*([u]_\alpha, [v]_\alpha) \geq \xi\},$$
$$S_{\xi,n} := S_\xi \cap (\frac{1}{n}, 1].$$

To show that $H^*([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$, it suffices to show that for each $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$, $S_{\xi,n}$ is a measurable set.

Since $v \in F^1_{USCG}(X)$, from Lemma 6.5 in [1] for each $k = 1, 2, \ldots$, there exist $\frac{1}{n} = \alpha_1^{(k)} < \cdots < \alpha_{l_k}^{(k)} = 1$ such that $w_v(\alpha_i^{(k)}, \alpha_{i+1}^{(k)}) \leq \frac{1}{k}$ for all $i = 1, \ldots, l_k - 1$.

Let $T_{k,i} := \{x :$ there exists $s \in S_\xi$ such that $\alpha_i^{(k)} < x \leq s \leq \alpha_{i+1}^{(k)}\}$. Put $T_k := \cup_{i=1}^{l_k-1} T_{k,i}$. We affirm that

(i) T_k is a measurable set,

(ii) $T_k \supseteq S_{\xi,n}$, and

(iii) $T_k \subseteq S_{\xi - \frac{1}{k}, n}$.

If $T_{k,i} \neq \emptyset$, then $T_{k,i}$ is an interval. Thus (i) is true. (ii) follows from the definition of T_k.

For each $i = 1, \ldots, l_k - 1$ and each $x \in T_{k,i}$, there exists an $s \in S_\xi$ such that $\alpha_i^{(k)} < x \leq s \leq \alpha_{i+1}^{(k)}$, and thus

$$\begin{aligned}
&H^*([u]_x, [v]_x) \\
&\geq H^*([u]_s, [v]_x) \\
&\geq H^*([u]_s, [v]_s) - H^*([v]_x, [v]_s) \\
&\geq \xi - 1/k.
\end{aligned}$$

Hence $T_k \subseteq S_{\xi - \frac{1}{k}}$. Clearly, $T_k \subseteq (\frac{1}{n}, 1]$. So (iii) is proved.

By affirmations (ii) and (iii), we have

$$S_{\xi,n} \subseteq \bigcap_{k=1}^{+\infty} T_k \subseteq \bigcap_{k=1}^{+\infty} S_{\xi - \frac{1}{k}, n} = S_{\xi,n}. \tag{8}$$

From affirmation (i), $\bigcap_{k=1}^{+\infty} T_k$ is measurable, and thus by (8), $S_{\xi,n} = \bigcap_{k=1}^{+\infty} T_k$ is measurable.

Step (II) $H^*([v]_\alpha, [u]_\alpha)$ is a measurable function of α on $[0,1]$.

The proof of Step (II) is similar to that of Step (I).

Let $\xi \in \mathbb{R}$ and let $n \in \mathbb{N}$. Define

$$S^{\xi} := \{\alpha \in [0,1] : H^*([v]_\alpha, [u]_\alpha) \geq \xi\},$$
$$S^{\xi,n} := S^{\xi} \cap (\frac{1}{n}, 1].$$

To show that $H^*([v]_\alpha, [u]_\alpha)$ is a measurable function of α on $[0,1]$, it suffices to show that for each $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$, $S^{\xi,n}$ is a measurable set.

Let $T^{k,i} := \{x :$ there exists $s \in S^{\xi}$ such that $\alpha_i^{(k)} < s \leq x \leq \alpha_{i+1}^{(k)}\}$. Put $T^k := \cup_{i=1}^{l_k-1} T^{k,i}$. We affirm that

(i′) T^k is a measurable set,

(ii′) $T^k \supseteq S^{\xi,n}$, and

(iii′) $T^k \subseteq S^{\xi-\frac{1}{k},n}$.

(i′) is true because if $T^{k,i} \neq \emptyset$, then $T^{k,i}$ is a point or an interval. (ii′) follows from the definition of T^k.

For each $i = 1, \ldots, l_k - 1$ and each $x \in T^{k,i}$, there exists an $s \in S^{\xi}$ such that $\alpha_i^{(k)} < s \leq x \leq \alpha_{i+1}^{(k)}$, and thus

$$\begin{aligned}
&H^*([v]_x, [u]_x) \\
&\geq H^*([v]_x, [u]_s) \\
&\geq H^*([v]_s, [u]_s) - H^*([v]_s, [v]_x) \\
&\geq \xi - 1/k.
\end{aligned}$$

Hence $T^k \subseteq S_{\xi-\frac{1}{k}}$. Clearly, $T^k \subseteq (\frac{1}{n}, 1]$. So (iii′) is proved.

From affirmations (ii′) and (iii′),

$$S^{\xi,n} \subseteq \bigcap_{k=1}^{+\infty} T^k \subseteq \bigcap_{k=1}^{+\infty} S^{\xi-\frac{1}{k},n} = S^{\xi,n}. \tag{9}$$

So by affirmation (i′) and (9), $S^{\xi,n} = \bigcap_{k=1}^{+\infty} T^k$ is measurable.

Step (III) $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.

Since that $H([u]_\alpha, [v]_\alpha) = \max\{H^*([u]_\alpha, [v]_\alpha), H^*([v]_\alpha, [u]_\alpha)\}$, then the desired result follows immediately from the fact that both $H^*([u]_\alpha, [v]_\alpha)$ and $H^*([v]_\alpha, [u]_\alpha)$ are measurable functions of α on $[0,1]$, which is proved in steps (I) and (II).

□

Remark 4.2. Theorem 4.1 is an improvement of Proposition 2.7. Since a singleton set is a compact set, Theorem 4.1 is also an improvement of Proposition 2.1.

Obviously, if $\xi \le 0$, then $S_\xi = S^\xi = [0,1]$ and $S_{\xi,n} = S^{\xi,n} = [\frac{1}{n}, 1]$.

Let (X, d_X) be a metric subspace of (Y, d_Y). To distinguish from the closure of S in (X, d_X), we use $\overline{S}^Y$ to denote the closure of S in (Y, d_Y).

For $u \in F^1_{USC}(X)$, define $u^Y \in F^1_{USC}(Y)$ as

$$[u^Y]_\alpha = \cap_{\beta<\alpha}\overline{[u]_\beta}^Y \text{ for } \alpha \in (0,1].$$

Note that $[u^Y]_\alpha = \cap_{\beta<\alpha}[u^Y]_\beta$ for all $\alpha \in (0,1]$, then by Theorem 2.9, u^Y is well-defined.

For each $u \in F^1_{USC}(X)$, define

$$\Gamma(u)^Y := \{\alpha \in (0,1] : [u^Y]_\alpha \supsetneqq \overline{[u]_\alpha}^Y\}.$$

If there is no confusion, we will write $\Gamma(u)^Y$ as $\Gamma(u)$ for simplicity.

We use H to denote the Hausdorff metric on $C(X)$ induced by d_X, and we also use H to denote the Hausdorff metric on $C(Y)$ induced by d_Y.

We will use the following Theorem 4.3 to improve Theorem 4.1 and Proposition 2.2.

Theorem 4.3. *Let (X, d_X) be a metric subspace of (Y, d_Y) and let $u, v \in F^1_{USC}(X)$. Then*
(i) $[u^Y]_\alpha \supseteq \overline{[u]_\alpha}^Y$ for all $\alpha \in (0,1]$, and $[u^Y]_0 = \overline{[u]_0}^Y$.
(ii) For each $\alpha \in [0,1] \setminus (\Gamma(u) \cup \Gamma(v))$,

$$H([u^Y]_\alpha, [v^Y]_\alpha) = H([u]_\alpha, [v]_\alpha).$$

(iii) The cardinality of $\Gamma(u)$ is less than the cardinality of $Y \setminus X$.

Proof. (i) follows from the definition of u^Y.

From (i) and the definition of $\Gamma(u)$, for each $\alpha \in [0,1] \setminus (\Gamma(u) \cup \Gamma(v))$,

$$H([u^Y]_\alpha, [v^Y]_\alpha) = H(\overline{[u]_\alpha}^Y, \overline{[v]_\alpha}^Y) = H([u]_\alpha, [v]_\alpha),$$

and thus (ii) is proved.

To show that (iii) is true, it suffices to construct an injection $j : \Gamma(u) \to Y \setminus X$.

Let $\gamma \in \Gamma(u)$. Then there is an $x_\gamma \in Y$ such that $x_\gamma \in [u^Y]_\gamma \setminus \overline{[u]_\gamma}^Y$. Define $j(\gamma) = x_\gamma$ for each $\gamma \in \Gamma(u)$. Since $x_\gamma \notin [u]_\gamma = \cap_{\beta<\gamma}[u]_\beta$, there is a $\beta < \gamma$ such that $x_\gamma \notin [u]_\beta$. On the other hand, since $x_\gamma \in [u^Y]_\gamma$, we have $x_\gamma \in \overline{[u]_\beta}^Y$. Thus $x_\gamma \in Y \setminus X$. Hence j is an function from $\Gamma(u)$ to $Y \setminus X$.

Let $\xi, \eta \in \Gamma(u)$ with $\xi < \eta$. Since $x_\xi \notin \overline{[u]_\xi}^Y$, then $x_\xi \notin [u^Y]_\lambda$ when $\lambda > \xi$. Hence $x_\xi \notin [u^Y]_\eta$. Notice that $x_\eta \in [u^Y]_\eta$, and therefore $x_\xi \neq x_\eta$. Thus j is an injection. So (iii) is proved.

□

Corollary 4.4. *Let (X, d_X) be a metric subspace of (Y, d_Y) and $Y \setminus X$ an at most countable set. Then for $u, v \in F^1_{USC}(X)$, $H([u^Y]_\alpha, [v^Y]_\alpha)$ is a measurable function of α on $[0,1]$ is equivalent to $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.*

Proof. By (ii), (iii) of Theorem 4.3, we have that $H([u^Y]_\alpha, [v^Y]_\alpha) = H([u]_\alpha, [v]_\alpha)$ on $[0,1]$ except at most countable $\alpha \in [0,1]$. Thus we obtain the desired result.

□

Let $S \subseteq \mathbb{R}^m$. We see $\mathbb{R}^m \setminus S$ as a metric subspace of $\mathbb{R}^m$.

Corollary 4.5. *Let S be an at most countable subset of $\mathbb{R}^m$. For $u, v \in F^1_{USC}(\mathbb{R}^m \setminus S)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.*

Proof. The desired result follows from Proposition 2.2 and Corollary 4.4.

□

Corollary 4.6. *Let (X, d_X) be a metric subspace of (Y, d_Y) and $Y \setminus X$ an at most countable set. Let $u, v \in F^1_{USC}(X)$. If $u^Y \in F^1_{USCG}(Y)$, then $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.*

Proof. The desired result follows from Theorem 4.1 and Corollary 4.4.

□

Remark 4.7. Clearly, if $u \in F^1_{USCG}(X)$, then $[u]_\alpha = [u^Y]_\alpha$ for $\alpha \in (0,1]$ and thus $u^Y \in F^1_{USCG}(Y)$. So Corollary 4.6 is an improvement of Theorem 4.1.

Corollary 4.5 is an improvement of Proposition 2.2.

Theorem 4.1 is the special case of Corollary 4.6 when $Y = X$. Proposition 2.2 is the special case of Corollary 4.5 when $S = \emptyset$.

In essence, contents including Theorem 4.3, Corollaries 4.4 and 4.5 have already been proved in chinaXiv:202108.00116v1, which is a previous version of this paper.

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